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The Greatest Common Divisor of Certain Sets of Binomial Coefficients

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A formula is obtained for the greatest common divisor of any number of consecutive terms in any given row of Pascal's triangle.

1. INTRODUCTION

Consider the Pascal triangle, written as an infinite lower triangular matrix

$$\begin{array}{llllllll}
\binom{0}{0} & & & & & & & \\
\binom{1}{0} & \binom{1}{1} & & & & 0 & & \\
\binom{2}{0} & \binom{2}{1} & \binom{2}{2} & & & & & \\
\cdots & & & & & & & \\
\binom{n}{0} & \binom{n}{1} & \binom{n}{2} & \cdots \binom{n}{r} & \cdots & \binom{n}{s} & \cdots & \binom{n}{n} \\
\cdots & & & & & & &
\end{array}$$

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Its rows have the following elementary property [3]: for $n \geqslant 2$,

$$\gcd\left\{\binom{n}{k} \mid k = 1,\dots, n-1\right\} = p \qquad \text{if } n = p^a, \tag{1}$$
$$= 1 \qquad \text{otherwise}$$

(p a prime and $a \geqslant 1$, an integer).

The problem which we discuss in this paper is that of obtaining a formula for the gcd of any number of consecutive terms in a given row of this matrix, say

$$d(n; r, s) := \gcd\left\{\binom{n}{k} \mid k = r,\dots, s\right\}, \tag{2}$$

where $n \geqslant r \geqslant 0$ and $s \geqslant r$.

We shall solve this problem completely. At the same time, we shall determine certain other sets of binomial coefficients chosen in the rows, columns, or diagonals of the Pascal matrix, whose gcd is equal to $d(n; r, s)$.

The main steps in the solution are as follows. We begin by proving a formula for $d(n; 1, s)$ in Chapter 2 (Theorem 1), and then consider the case $r \geqslant 2$. As we shall see in Chapter 5 (Theorem 2), the formula for $d(n; r, s)$ is relatively simple when s is sufficiently large with respect to r:

$$d(n; r+1, s) = \prod_{l=0}^{r} d(n-l; 1, s) \qquad \text{if } s \geqslant 2r.$$

If $r < s < 2r$, the right side of this identity must be multiplied by an additional factor which is a product of certain primes from the interval $[2, r]$; this factor is determined in Chapters 6 and 7 (Theorem 3).

Throughout this article, we denote the gcd of integers $a_1,\dots, a_n$ by $(a_1,\dots, a_n)$.

For a discussion of related problems, see [1 and 4]; the latter includes an extensive bibliography.

2. The Case $r = 1$

If $d(n; r, s)$ is defined as in (2), and if $s \geqslant n$ or $r = 0$, then $d(n; r, s) = 1$ since $\binom{n}{0} = \binom{n}{n} = 1$. Accordingly we shall assume, whenever it is convenient, that $n > s \geqslant r \geqslant 1$.

We begin by proving

Lemma 1. *For $n \geqslant s \geqslant 1$ we have*

$$\left(d(n; 1, s), \binom{n-1}{s}\right) = 1. \tag{3}$$

Proof. From the recurrence formula

$$\binom{n}{m}=\binom{n-1}{m-1}+\binom{n-1}{m} \tag{4}$$

one easily deduces the identity

$$\sum_{m=0}^{s}(-1)^m\binom{n}{m}=(-1)^s\binom{n-1}{s},$$

which implies that

$$d(n;1,s)\left|\binom{n-1}{s}+(-1)^{s+1};\right.$$

(3) follows immediately.

We can now prove our basic formula, of which (1) is a particular case.

THEOREM 1. *For $n\geqslant s\geqslant 1$, we have*

$$d(n;1,s)=\frac{n}{[1^{\varepsilon_1(n)},2^{\varepsilon_2(n)},3^{\varepsilon_3(n)},...,s^{\varepsilon_s(n)}]}, \tag{5}$$

where

$$\begin{aligned}\varepsilon_j(m)&=1 &&\text{if } j\mid m,\\ &=0 &&\text{otherwise,}\end{aligned}$$

and where the square brackets in the denominator denote the lcm *of the integers they enclose. (The denominator on the right side of* (5) *is the* lcm *of the positive divisors of n that do not exceed s.)*

Proof. For simplicity of notation we write ε_j in place of $\varepsilon_j(n)$. The proof is by induction on s. For $s=1$, (5) asserts that $d(n;1,1)=n$, which is obviously true.

Now assume that (5) holds with $s-1$ instead of s, for some s with $2\leqslant s\leqslant n$. We write

$$d(n;1,s)=\left(d(n;1,s-1),\binom{n}{s}\right) \tag{6}$$

and distinguish 2 cases, according to whether s divides n or not.

(a) If $s\mid n$, we use (6) and Lemma 1 to get

$$d(n;1,s)=\left(d(n;1,s-1),\frac{n}{s}\binom{n-1}{s-1}\right)=\left(d(n;1,s-1),\frac{n}{s}\right).$$

Then (5) follows on using the induction hypothesis, since $\varepsilon_s = 1$ and since

$$\left(\frac{n}{d_1}, \frac{n}{d_2}\right) = \frac{n}{[d_1, d_2]}$$

if d_1 and d_2 divide n $(n \geqslant 1)$.

(b) If $s \nmid n$, then

$$[1^{\varepsilon_1}, 2^{\varepsilon_2}, ..., (s-1)^{\varepsilon_{s-1}}] = [1^{\varepsilon_1}, 2^{\varepsilon_2}, ..., s^{\varepsilon_s}],$$

since $\varepsilon_s = 0$; (6) and the induction hypothesis then yield

$$d(n; 1, s) = \left(\frac{n}{[1^{\varepsilon_1}, 2^{\varepsilon_2}, ..., s^{\varepsilon_s}]}, \binom{n}{s}\right).$$

From this, (5) follows on observing that (for any n, s with $n \geqslant s \geqslant 1$)

$$(n, s) \mid [1^{\varepsilon_1}, 2^{\varepsilon_2}, ..., s^{\varepsilon_s}] \qquad \text{and} \qquad n\binom{n-1}{s-1} = s\binom{n}{s},$$

whence

$$\frac{n}{[1^{\varepsilon_1}, 2^{\varepsilon_2}, ..., s^{\varepsilon_s}]} \,\Big|\, \frac{n}{(n, s)} \qquad \text{and} \qquad \frac{n}{(n, s)} \,\Big|\, \binom{n}{s}.$$

This concludes the proof of Theorem 1.

Remark. Using the prime number theorem, one can deduce from Theorem 1 that for any δ, $0 < \delta < 1$, there exists an $n_0(\delta)$ such that $d(n; 1, s) > n^{1-\delta}$ if $n \geqslant n_0(\delta)$ and $s \leqslant (\delta/2) \log n$.

3. Triangles

Let $n \geqslant s > r \geqslant 0$, and consider the binomial coefficients $\binom{m}{l}$ with $n \leqslant m \leqslant n - r + s$ and $m - n + r \leqslant l \leqslant s$, arrayed as in the matrix of Chapter 1. They form a triangle, say $T(n; r, s)$, with vertices $\binom{n}{r}$, $\binom{n}{s}$ and $\binom{n-r+s}{s}$, right-angled at $\binom{n}{s}$. We will speak of rows, columns, and diagonals of this triangle as we would for the matrix itself, and number these lines starting from $\binom{n}{r}$. For example, the vth column of $T(n; r, s)$ consists of the binomial coefficients $\binom{n+k}{r+v}$, $k = 0, ..., v$ $(0 \leqslant v \leqslant s - r)$.

We shall show that $d(n; r, s)$, the gcd of the coefficients in the first row of $T(n; r, s)$, is equal to the gcd of certain other sets of $s - r + 1$ binomial coefficients taken from the same triangle.

The simplest case is that in which $s-r=1$. Since $(a, b)=(a, a+b)=(b, a+b)$ we have by (4),

$$\left(\binom{m}{l-1}, \binom{m}{l}\right)=\left(\binom{m}{l-1}, \binom{m+1}{l}\right) \tag{7}$$

and

$$\left(\binom{m}{l-1}, \binom{m}{l}\right)=\left(\binom{m}{l}, \binom{m+1}{l}\right). \tag{8}$$

We generalize (7) and (8) by proving

PROPOSITION 1. *If $n \geqslant s > r \geqslant 0$ and if $r+v \leqslant j_v \leqslant s$ for $v=0, 1,..., s-r$, then*

$$d(n; r, s)=\left(\binom{n}{j_0}, \binom{n+1}{j_1},..., \binom{n+v}{j_v},..., \binom{n-r+s}{j_{s-r}}\right). \tag{9}$$

($d(n; r, s)$ is equal to the gcd *of any set of $s-r+1$ binomial coefficients, chosen one in each row of $T(n; r, s)$.)*

Proof. We argue by induction on $s-r$. For $s-r=1$, (9) follows from (7) and (8). Now assume that (9) holds for all $T(n'; r', s')$ with $s'-r'=d-1$ (some $d \geqslant 2$), and consider a $T(n; r, s)$ with $s-r=d$. By applying the induction hypothesis to $T(n+1; r+1, s)$, we see that it suffices to prove that

$$\left(\binom{n}{k}, d(n+1; r+1, s)\right)=d(n; r, s) \qquad \text{if } r \leqslant k \leqslant s.$$

Now for $r<k \leqslant s$ we have

$$\left(\binom{n}{k}, d(n+1; r+1, s)\right)=\left(\binom{n}{r}, d(n+1; r+1, s)\right)$$

by repeated application of the case $s-r=1$. And by repeated application of (7),

$$\left(\binom{n}{r}, d(n+1; r+1, s)\right)=d(n; r, s), \tag{10}$$

thus proving Proposition 1.

The same type of proof will establish the following generalizations of (7) and of (8), respectively, PROPOSITION 2. *If $n \geqslant s > r \geqslant 0$ and if $0 \leqslant i_\nu \leqslant \nu$ for $\nu = 0, 1, \ldots, s-r$, then*

$$d(n; r, s) = \left(\binom{n}{r}, \binom{n+i_1}{r+1}, \ldots, \binom{n+i_\nu}{r+\nu}, \ldots, \binom{n+i_{s-r}}{s} \right). \tag{11}$$

($d(n; r, s)$ is equal to the gcd *of any set of $s-r+1$ binomial coefficients, taken one in each column of $T(n; r, s)$.)*

PROPOSITION 3. *If $n \geqslant s > r \geqslant 0$ and if $r+\nu \leqslant k_\nu \leqslant s$ for $\nu = 0, 1, \ldots, s-r$, then*

$$d(n; r, s) = \left(\binom{n-r+k_0}{k_0}, \ldots, \binom{n-r+k_\nu-\nu}{k_\nu}, \ldots, \binom{n}{s} \right). \tag{12}$$

($d(n; r, s)$ is equal to the gcd *of any set of $s-r+1$ binomial coefficients, chosen one in each diagonal of $T(n; r, s)$.)*

We call the first row, first diagonal, and last column of $T(n; r, s)$ its *sides.* The following result contains Propositions 1, 2, and 3 as particular cases; they are used in its proof (which we omit).

PROPOSITION 4. *Let C_ν, $\nu = 0, \ldots, s-r$, be binomial coefficients chosen in $T(n; r, s)$. If some side of this triangle contains exactly one of the C_ν; if on removing this side the remaining triangle has the same property, and so on, until a single binomial coefficient is left, itself one of the C_ν, then the* gcd *of $C_0, \ldots, C_{s-r}$ is equal to $d(n; r, s)$.*

We use Theorem 1 and Proposition 2 to establish

LEMMA 2. *Let p be a prime and n and α positive integers; let $n \geqslant p^\alpha$. Write $n = kp^\beta + r$ with $p \nmid k$, $0 \leqslant r \leqslant p^\alpha - 1$ and $\beta \geqslant \alpha$. Then*

$$p^{\beta-\alpha} \,\Big\|\, \binom{n}{p^\alpha}. \tag{13}$$

(r is the remainder and $kp^{\beta-\alpha}$ the quotient, when n is divided by p^α.)

Proof. It follows from (5) that if $p^\beta \,\|\, m$ and $s = p^\alpha$ with $\beta \geqslant \alpha \geqslant 1$, then

$$p^{\beta-\alpha} \,\|\, d(m; 1, s) \tag{14}$$

and

$$p^{\beta-(\alpha-1)} \,\|\, d(m; 1, s-1). \tag{15}$$

Proposition 2 and (14) imply that

$$p^{\beta-\alpha}\left\|\left(d(m; 1, s-1), \binom{m+r}{s}\right)\right., \qquad r=0, 1,..., s-1;$$

from this and (15) we deduce that $p^{\beta-\alpha}\|\binom{m+r}{s}$, which is (13) if $m=kp^{\beta}$.

Lemma 2 could also be proved by appealing to a theorem of Glaisher's [2] according to which the number of times a prime p divides $\binom{n}{l}$ is equal to the number of borrows when the subtraction $n-l$ is done in base p.

4. Divisors of $d(n; r, s)$

After Theorem 1 and Lemma 2, the next step towards our formula for $d(n; r, s)$ is

Proposition 5. *For $n>l_2$ and $0\leqslant l_1<l_2\leqslant s$, we have*

$$(d(n-l_1; 1, s), d(n-l_2; 1, s))=1.$$

Proof. It clearly suffices to show that

$$(d(n; 1, s), d(n-l; 1, s))=1 \tag{16}$$

for all $n>l$ and all l with $1\leqslant l\leqslant s$. We may assume that $n>s$.

By (10), $d(n-l; 1, s)$ divides $d(n-l+1; 2, s)$, hence by induction divides $d(n-l+k; k+1, s)$ for $1\leqslant k+1\leqslant s$. Taking $k=l-1$, we see that $d(n-l; 1, s)\,|\,d(n-1; l, s)\,|\binom{n-1}{s}$, if $1\leqslant l\leqslant s$. Therefore, the left side of (16) divides $(d(n; 1, s), \binom{n-1}{s})$; we conclude the proof by appealing to Lemma 1.

Proposition 6. *If $n\geqslant s\geqslant r\geqslant 1$, then*

$$\prod_{l=0}^{r-1} d(n-l; 1, s)\,\Big|\, d(n; r, s). \tag{17}$$

Proof. Because of Proposition 5, it suffices to show that $d(n-l; 1, s)$ divides $d(n; r, s)$ for $0\leqslant l\leqslant r-1$. In the proof of Proposition 5, we showed that $d(n-l; 1, s)$ divides $d(n-l+k; k+1, s)$ for $1\leqslant k+1\leqslant s$. Since $l+1\leqslant r\leqslant s$, we may take $k=l$: $d(n-l; 1, s)\,|\,d(n; l+1, s)$ for $0\leqslant l\leqslant s-1$. Since obviously $d(n; l+1, s)\,|\,d(n; r, s)$ for $0\leqslant l\leqslant r-1$, the proof is complete.

5. The Case $s \geqslant 2r$

In this chapter, we prove

Theorem 2. *For* $n \geqslant s \geqslant 2r \geqslant 2$,

$$d(n; r+1, s) = \prod_{l=0}^{r} d(n-l; 1, s). \tag{18}$$

We introduce some notation. As before, p always denotes a prime. For rational $a \neq 0$, let N be the integer such that $p^N \| a$; then we write $\operatorname{ord}_p(a) := N$ and $a_p := p^N$. If a and b are rational ($a \neq 0$ and $b \neq 0$), we write $a \sim_p b$ if $a_p = b_p$, and $a \prec_p b$ if $a_p \leqslant b_p$; when there is no ambiguity concerning the choice of p we simply write $a \sim b$, respectively $a \prec b$.

The following lemma is required several times in the sequel.

Lemma 3. *Let* $n \geqslant m \geqslant r+1 \geqslant 1$, *and* $p \mid n$. *Set* $n = p\nu$, $\rho = [r/p]$ *and* $\mu = [m/p]$. *Then*

(i) $d(n; r+1, m) \sim_p p(\nu - \mu)\binom{\nu}{\mu}$ *if* $\mu p < r+1$,

(ii) $d(n; r+1, m) \sim_p d(\nu; \rho+1, \mu)$ *if* $\mu p \geqslant r+1$.

Proof. Since

$$\binom{n}{a} = \frac{n}{a} \cdot \frac{n-1}{1} \cdot \frac{n-2}{2} \cdots \frac{n-(a-1)}{a-1},$$

it is clear that for $n = p\nu$,

$$\binom{n}{a} \sim \frac{p\nu}{a} \frac{\nu-1}{1} \cdot \frac{\nu-2}{2} \cdots \frac{\nu-\alpha}{\alpha}, \qquad \alpha = \left[\frac{a-1}{p}\right]. \tag{19}$$

Now if $r+1 \leqslant a \leqslant m$ and $\mu p < r+1$, then $\mu p < a < (\mu+1)p$, whence $\alpha = \mu$ and $p \nmid a$, so that

$$\binom{n}{a} \sim p\nu \frac{\nu-1}{1} \cdot \frac{\nu-2}{2} \cdot \cdots \cdot \frac{\nu-\mu}{\mu} = p(\nu-\mu)\binom{\nu}{\mu},$$

as required in (i).

To prove (ii) we begin by showing that if k is an integer and $r+1 \leqslant kp$, then

$$d(n; r+1, kp) \sim d(\nu; \rho+1, k). \tag{20}$$

Indeed,

$$(d(n; r+1, kp))_p = \min_{r+1 \leqslant a \leqslant kp} \binom{n}{a}_p,$$

and (19) shows that for fixed n and α, $\binom{n}{a}_p$ is smallest when a_p is largest. And among all a such that $\alpha p < a \leqslant (\alpha+1)p$, a_p is largest when $a = (\alpha+1)p$. Since α varies from ρ to $k-1$ when a varies from $r+1$ to kp, we have from (19),

$$\begin{aligned} d(n; r+1, kp) &\sim \min_{\rho \leqslant \alpha \leqslant k-1} \left(\frac{\nu}{\alpha+1} \cdot \frac{\nu-1}{1} \cdot \cdots \cdot \frac{\nu-\alpha}{\alpha} \right)_p \\ &= \min_{\rho \leqslant \alpha \leqslant k-1} \binom{\nu}{\alpha+1}_p \sim d(\nu; \rho+1, k). \end{aligned}$$

Thus (20) is verified, and with it (ii) for $m = \mu p$. Finally, if $r+1 \leqslant \mu p < m$ we have

$$d(n; r+1, m) = (d(n; r+1, \mu p),\ d(n; \mu p+1, m))$$

so that by (20) and (i),

$$d(n; r+1, m) \sim \left(d(\nu; \rho+1, \mu),\ p(\nu-\mu)\binom{\nu}{\mu} \right) \sim d(\nu; \rho+1, \mu),$$

since $p(\nu-\mu)\binom{\nu}{\mu} \succ \binom{\nu}{\mu} \succ d(\nu; \rho+1, \mu)$. The lemma is now proved in both cases.

After this preparation we pass to the

Proof of Theorem 2. Because of Proposition 6, it suffices to establish that

$$d(n; r+1, s) \,\Big|\, \prod_{l=0}^{r} d(n-l; 1, s) \qquad \text{if } n \geqslant s \geqslant 2r \geqslant 2.$$

We do this by a double induction argument. First we show that if the theorem holds for some s, it also holds for all larger s (n and r being fixed). Then we show that Theorem 2 is true for $s = 2r$, $r \geqslant 1$, if it is true for $s = 2$, $r = 1$.

(a) *Induction on s.* Fix n and r, and suppose (18) is true for $s = s_0 - 1$ (some $s_0 \geqslant r+2$). By (5),

$$\begin{aligned} d(n-l; 1, s-1) &= pd(n-l; 1, s) && \text{if } s = p^a \text{ and } p^a \mid n-l, \\ &= d(n-l; 1, s) && \text{otherwise.} \end{aligned} \tag{21}$$

By the induction hypothesis,

$$d(n; r+1, s_0) = \left(\prod_{l=0}^{r} d(n-l; 1, s_0 - 1), \binom{n}{s_0} \right). \tag{22}$$

We distinguish the 2 cases envisaged in (21). If $s_0 = p^a$ and $p^a \nmid n-l$ for $0 \leqslant l \leqslant r$, or if s_0 is not of the form p^a, (21) and (22) yield

$$d(n; r+1, s_0) = \left(\prod_{l=0}^{r} d(n-l; 1, s_0), \binom{n}{s_0} \right) = \prod_{l=0}^{r} d(n-l; 1, s_0),$$

for by (17) this product divides $d(n; r+1, s_0)$, itself a divisor of $\binom{n}{s_0}$.

Otherwise, $s_0 = p^a$ and $p^a \mid n-l$ for some l, $0 \leqslant l \leqslant r$ (since $0 \leqslant l \leqslant r$ and $p^a = s_0 > r+1$, there can be at most one l for which $p^a \mid n-l$). Then, from (21) and (22),

$$d(n; r+1, p^a) = \left(p \prod_{l=0}^{r} d(n-l; 1, p^a), \binom{n}{p^a} \right) \tag{23}$$

$$= \prod_{l=0}^{r} d(n-l; 1, p^a),$$

where the last equality can be justified as follows. In (23), the product divides the binomial coefficient, as in the preceding case. Further, for the prime p,

$$\binom{n}{p^a} \sim \prod_{l=0}^{r} d(n-l; 1, p^a).$$

Indeed, $n - l = p^b k$ (with $p \nmid k$, some b and l, $b \geqslant a$, $0 \leqslant l \leqslant r \leqslant p^a - 1$), so that $p^{b-a} \| \binom{n}{p^a}$ by (13); by (14), $p^{b-a} \| d(n-l; 1, p^a)$.

(b) *Induction on r.* We wish to show that

$$d(n; r+1, 2r) \,\Big|\, \prod_{l=0}^{r} d(n-l; 1, 2r) \qquad \text{for } r \geqslant 1.$$

This is true for $r = 1$ since by (5),

$$d(n; 2, 2) = \frac{1}{2} n(n-1) = \frac{n}{2^{\varepsilon_2(n)}} \cdot \frac{n-1}{2^{\varepsilon_2(n-1)}} = d(n; 1, 2)\, d(n-1; 1, 2).$$

Proposition 6 and part (a) of the present proof allow us to formulate the induction hypothesis as follows: we assume that for some $r \geqslant 2$,

$$d(n; m+1, s) = \prod_{l=0}^{m} d(n-l; 1, s) \qquad \text{for } 1 \leqslant m \leqslant r-1, \text{ if } n \geqslant s \geqslant 2m. \tag{24}$$

Under this assumption we prove that (24) also holds for $m = r$, by showing that

$$p^\alpha \mid d(n; r+1, 2r) \Rightarrow p^\alpha \,\Big|\, \prod_{l=0}^{r} d(n-l; 1, 2r). \tag{25}$$

For this we distinguish 3 cases: $p \nmid n-r$, $p \nmid n$, and $p \mid (n, r)$.

(b_1) If $p \nmid n-r$, then

$$\binom{n}{r+1} \prec (r+1)\binom{n}{r+1} = (n-r)\binom{n}{r} \sim \binom{n}{r},$$

whence

$$d(n; r+1, s) \sim d(n; r, s) \qquad \text{if } p \nmid n-r. \tag{26}$$

On setting $s = 2r$ in (26) and applying the induction hypothesis (24) with $m = r-1$ and $s = 2m+2$ to $d(n; r, 2r)$, we see that p^α divides $\prod_{l=0}^{r-1} d(n-l; 1, 2r)$, and (25) must hold.

(b_2) If $p \nmid n$, we use the relation

$$d(n; r+1, s) \sim d(n-1; r, s) \qquad \text{if } p \nmid n, \tag{27}$$

which we establish by showing that (for all n, r, s)

$$d(n-1; r, s) \mid d(n; r+1, s) \mid nd(n-1; r, s). \tag{28}$$

Indeed, the first part of (28) is implied by (10); for the second we combine the identity $k\binom{n}{k} = n\binom{n-1}{k-1}$, which implies that $d(n; r+1, s) \mid nd(n-1; r, s-1)$, and (4) which shows that $(\binom{n}{s}, n\binom{n-1}{s-1})$ divides $n\binom{n-1}{s}$.

Then (25) follows from (27) with $s = 2r$, and (24) with $n-1$ in place of n and $m = r-1$, $s = 2m+2$.

(b_3) If $p \mid (n, r)$, Lemma 3(ii) can be applied to $d(n; r+1, 2r)$: since $m = 2r$ we have $\mu p = m = 2r \geqslant r+1$. By this lemma, $d(n; r+1, 2r) \sim d(\nu; \rho+1, 2\rho)$ with $p\nu = n$ and $p\rho = r$. And (24) applies to $d(\nu; \rho+1, 2\rho)$, since $\rho \leqslant \frac{1}{2}r \leqslant r-1$. Therefore

$$d(n; r+1, 2r) \sim \prod_{\lambda=0}^{\rho} d(\nu - \lambda; 1, 2\rho),$$

so that p^α divides one of the (pairwise relatively prime) factors $d(\nu-\lambda; 1, 2\rho)$, $0 \leqslant \lambda \leqslant \rho$. But again by Lemma 3(ii), $d(\nu-\lambda; 1, 2\rho) \sim d(n - \lambda p; 1, 2r)$; since $0 \leqslant \lambda p \leqslant p\rho = r$, we have shown that p^α divides one of the factors on the right side of (24).

This concludes the proof of Theorem 2.

6. The Case $s<2r$: Recurrence Formulas

We know that $\prod_{l=0}^{r} d(n-l; 1, s)$ divides $d(n; r+1, s)$, and that the two are equal if $s \geqslant 2r$. For $r+1 \leqslant s$, we set $s=2r-\delta$ $(\delta \leqslant r-1)$ and define an integer $E_\delta(n, r)$ by

$$d(n; r+1, 2r-\delta)=E_\delta(n, r) \prod_{l=0}^{r} d(n-l; 1, 2r-\delta); \tag{29}$$

for $\delta \leqslant 0$, (29) holds with $E_\delta(n, r)=1$.

The prime divisors of $E_\delta(n, r)$ divide $d(n; r+1, s)$; in particular, it suffices to consider the primes that divide $\binom{n}{r+1}$.

In this chapter we prove 5 propositions which, for a given p, reduce the determination of $\operatorname{ord}_p(E_\delta(n, r))$ to that of $\operatorname{ord}_p(E_{\delta'}(n' r'))$, with $n'+r'+\delta'<n+r+\delta$. This yields 5 recurrence relations which we solve in the next chapter to obtain a formula for $E_\delta(n, r)$.

We consider the following cases, which we analyze in Propositions 7 through 11 (in all cases, we assume $1 \leqslant \delta \leqslant r-1$):

Case 1. $p \nmid n-r$.

Case 2. $p \nmid n$.

Case 3. $p \mid (n, r)$, with the subcases

(a) $p \mid (n, r, \delta)$

(b) $p \mid (n, r)$, $p \nmid \delta$ and $r>p+\delta$

(c) $p \mid (n, r)$, $p \nmid \delta$ and $\delta<r<p+\delta$.

Proposition 7 (Case 1). *If $p \nmid n-r$, then*

$$E_\delta(n, r) \sim E_{\delta-2}(n, r-1). \tag{30}$$

Proof. We set $s=2r-\delta$ in (26) and expand the right side according to (29):

$$\begin{aligned} d(n; r+1, 2r-\delta) &\sim d(n; r, 2r-\delta) \\ &= E_{\delta-2}(n, r-1) \prod_{l=0}^{r-1} d(n-l; 1, 2r-\delta). \end{aligned}$$

Hence

$$d(n; r+1, 2r-\delta) \sim E_{\delta-2}(n, r-1) \prod_{l=0}^{r} d(n-l; 1, 2r-\delta),$$

because $d(n-r; 1, 2r-\delta) \sim 1$ if $p \nmid n-r$. On comparing with (29), we get (30).

PROPOSITION 8 (Case 2). *If $p\nmid n$, we have*

$$E_\delta(n, r) \sim E_{\delta-2}(n-1, r-1). \tag{31}$$

Proof. We use (27) with $s = 2r - \delta$, and proceed as in the proof of Proposition 7.

For the remaining cases we appeal to Lemma 3.

PROPOSITION 9 (Case 3(b)). If $p \mid (n, r)$, $p \nmid \delta$, *and* $r > p + \delta$, *then*

$$E_\delta(n, r) \sim E_{\delta_1+1}(n_1, r_1), \tag{32}$$

where $n = pn_1$, $r = pr_1$, *and* $\delta_1 = [\delta/p]$.

Proof. Lemma 3(ii) applies to $d(n; r+1, 2r-\delta)$, since $\mu p \geqslant r+1$ in Case 3(b). Indeed, $\mu p \geqslant r+1$ is equivalent to $\mu \geqslant r_1 + 1$ since μ and r_1 are integers; similarly, $r > \delta + p$ implies $r_1 \geqslant \delta_1 + 2$. And $\mu = [(2r-\delta)/p] = 2r_1 - \delta_1 - 1$ since $p \nmid \delta$. By Lemma 3 then, $d(n; r+1, 2r-\delta) \sim d(n_1; r_1+1, 2r_1-\delta_1-1)$ whence with (29), and a second application of Lemma 3(ii),

$$\begin{aligned} d(n; r+1, 2r-\delta) &\sim E_{\delta_1+1}(n_1, r_1) \prod_{\lambda=0}^{r_1} d(n_1-\lambda; 1, 2r_1-\delta_1-1) \\ &\sim E_{\delta_1+1}(n_1, r_1) \prod_{\lambda=0}^{r_1} d(n-\lambda p; 1, 2r-\delta). \end{aligned}$$

Hence, since $d(n-l; 1, 2r-\delta) \sim 1$ if $p \nmid n-l$,

$$d(n; r+1, 2r-\delta) \sim E_{\delta_1+1}(n_1, r_1) \prod_{l=0}^{r} d(n-l; 1, 2r-\delta),$$

and (32) follows.

PROPOSITION 10 (Case 3(c)). *If $p \mid (n, r)$, $p \nmid \delta$, and $\delta < r < p + \delta$, then*

$$E_\delta(n, r) \sim \frac{p(n_1 - r_1)}{d(n_1 - r_1; 1, r_1)} E_{r_1-2}(n_1, r_1 - 1), \tag{33}$$

where $n = pn_1$ *and* $r = pr_1$.

Proof. Lemma 3(i) applies to $d(n; r+1, 2r-\delta)$, because $\mu p < r+1$ in Case 3(c): $\delta < r < p + \delta$ implies $0 < r_1 - \delta/p < 1$, so that $\mu = [(2r-\delta)/p] = [r_1 + (r_1 - \delta/p)] = r_1$ and $\mu p = pr_1 = r$. By the lemma we know that

$$d(n; r+1, 2r-\delta) \sim p(n_1 - r_1)\binom{n_1}{r_1} = p(n_1 - r_1)\, d(n_1; r_1, r_1),$$

whence by (29),

$$d(n; r+1, 2r-\delta) \sim \frac{p(n_1 - r_1)}{d(n_1 - r_1; 1, r_1)} E_{r_1 - 2}(n_1, r_1 - 1) \prod_{\lambda=0}^{r_1} d(n_1 - \lambda; 1, r_1).$$

Since $d(n_1 - \lambda; 1, r_1) \sim d(n - \lambda p; 1, 2r - \delta)$ by Lemma 3(ii), the proof can be concluded as in the previous case.

PROPOSITION 11 (Case 3(a)). *If* $p \mid (n, r, \delta)$ *and* $n = pn_1$, $r = pr_1$, $\delta = p\delta_1$, *then*

$$E_\delta(n, r) \sim E_{\delta_1}(n_1, r_1). \tag{34}$$

Proof. By (20) we have $d(n; r+1, 2r-\delta) \sim d(n_1; r_1 + 1, 2r_1 - \delta_1)$, and $d(n - \lambda p; 1, 2r - \delta) \sim d(n_1 - \lambda; 1, 2r_1 - \delta_1)$ if λ is an integer; one then proceeds as in the proof of Proposition 9.

7. THE FORMULA FOR $E_\delta(n, r)$

Let $E_\delta(n, r)$ be defined by (29), with $n \geqslant r+1$ and $\delta \leqslant r-1$. In this chapter we obtain the representation of $E_\delta(n, r)$ as a product of primes.

THEOREM 3. *We have*

$$E_\delta(n, r) = \prod_{\substack{2 \leqslant m \leqslant r \\ m - \{r\}_m \geqslant r+1-\delta}} \lambda(m)^{e_m(n,r)}, \tag{35}$$

where $\{r\}_m$ *denotes the least residue of* r *modulo* m $(0 \leqslant \{r\}_m \leqslant m-1)$,

$$\lambda(m) := \begin{matrix} p & \text{if } m = p^k \ (k \geqslant 1), \\ 1 & \text{otherwise}, \end{matrix}$$

and

$$e_m(n, r) = \sum_{i=0}^{\{r\}_m} \varepsilon_m(n - i) \tag{36}$$

with $\varepsilon_m(n-i)$ *defined as in Theorem* 1.

We need a lemma, which provides another expression for the right side of (35).

LEMMA 4. *Let $F_\delta(n, r)$ be the product on the right side of* (35), *and let* $n \geqslant r+1$ *and* $\delta \leqslant r-1$. *Then,*

$$F_\delta(n, r) = \prod_{\substack{2 \leqslant m \leqslant r \\ m - \{r\}_m \geqslant r+1-\delta \\ \{n\}_m \leqslant \{r\}_m}} \lambda(m). \tag{37}$$

Proof. Since $0 \leqslant \{r\}_m \leqslant m-1$, at most one of the $\varepsilon_m(n-i)$ in (36) is equal to 1. This occurs if and only if there exists an integer i, $0 \leqslant i \leqslant \{r\}_m$, such that $m \mid n-i$, i.e., if and only if $\{n\}_m \leqslant \{r\}_m$.

We are now in a position to prove Theorem 3.

Proof of Theorem 3. When $\delta \leqslant 0$, we have $F_\delta(n, r) = 1$ (empty product) and $E_\delta(n, r) = 1$ (Theorem 2). Hence, to prove Theorem 3 it suffices to verify that $F_\delta(n, r)$ satisfies the same recurrence relations as $E_\delta(n, r)$. We consider the same cases as in Propositions 7 through 11.

Case 1. We must show that

$$F_\delta(n, r) \sim F_{\delta-2}(n, r-1) \qquad \text{if } p \nmid n-r. \tag{38}$$

It follows from (37) that

$$F_\delta(n, r) \sim p^{N_1} \qquad \text{and} \qquad F_{\delta-2}(n, r-1) \sim p^{N_2},$$

where

$$N_1 = \#\{s \geqslant 1 \mid p^s \leqslant r, p^s - \{r\}_{p^s} \geqslant r+1-\delta, \text{ and } \{n\}_{p^s} \leqslant \{r\}_{p^s}\} \tag{39}$$

and

$$N_2 = \#\{s \geqslant 1 \mid p^s \leqslant r-1, p^s - \{r-1\}_{p^s} \geqslant r+2-\delta, \text{ and } \{n\}_{p^s} \leqslant \{r-1\}_{p^s}\}.$$

In Case 1, the conditions for N_1 imply that $p^s \nmid r$: $\{n\}_{p^s} \neq \{r\}_{p^s}$ because $p \nmid n-r$, whence $\{r\}_{p^s} \geqslant 1$ by the last inequality in (39). The conditions for N_2 have the same implication: $p^s - \{r-1\}_{p^s} \geqslant r+2-\delta \geqslant 3$, whereas $\{r-1\}_{p^s} = p^s - 1$ if $p^s \mid r$.

Now when $p^s \nmid r$, we have $p^s \leqslant r$ if and only if $p^s \leqslant r-1$; $p^s \nmid r$ is equivalent to $\{r-1\}_{p^s} = \{r\}_{p^s} - 1$; and as already observed, $\{n\}_{p^s} \neq \{r\}_{p^s}$ in Case 1. It follows that $N_1 = N_2$, and (38) holds as asserted.

Case 2. Here one proves that $N_1 = N_3$ if $p \nmid n$, where N_1 is as in (39) and

$$N_3 = \#\{s \geqslant 1 \mid p^s \leqslant r-1, p^s - \{r-1\}_{p^s} \geqslant r+2-\delta,$$
$$\text{and } \{n-1\}_{p^s} \leqslant \{r-1\}_{p^s}\}.$$

The details are similar to those of Case 1.

Case 3(a). If $n=pn_1$, $r=pr_1$, and $\delta=p\delta_1$, we must show that $N_1=N_4$, with N_1 as in (39) and

$$N_4=\#\{s\geqslant 1\,|\,p^s\leqslant r_1, p^s-\{r_1\}_{p^s}\geqslant r_1+1-\delta_1, \text{ and } \{n_1\}_{p^s}\leqslant\{r_1\}_{p^s}\}.$$

But $s=1$ is not counted in N_1 when $p\,|\,(r,\delta)$; for if it were we would have $p=p-\{r\}_p\geqslant r+1-\delta=p(r_1-\delta_1)+1>p$, since $r>\delta$ and consequently $r_1>\delta_1$. Hence, in Case 3(a),

$$N_1=\#\{s\geqslant 2\,|\,p^s\leqslant r, p^s-\{r\}_{p^s}\geqslant r+1-\delta, \text{ and } \{n\}_{p^s}\leqslant\{r\}_{p^s}\}.$$

Now if k is a positive integer and $p\,|\,k$, say $k=pk_1$, then

$$\{k\}_{p^s}=p\{k_1\}_{p^{s-1}}. \tag{40}$$

Therefore,

$$\begin{aligned} N_1&=\#\{s\geqslant 2\,|\,p^{s-1}\leqslant r_1, p^{s-1}-\{r_1\}_{p^{s-1}}\geqslant r_1+1-\delta_1,\\ &\qquad\text{and } \{n_1\}_{p^{s-1}}\leqslant\{r_1\}_{p^{s-1}}\}\\ &=N_4. \end{aligned}$$

Case 3(b). We now show that

$$F_\delta(n,r)\sim F_{\delta_1+1}(n_1,r_1)\qquad \text{if } p\,|\,(n,r), p\nmid\delta, \text{ and } r>p+\delta$$

(here, $n=pn_1$, $r=pr_1$, and $\delta_1=[\delta/p]$). The proof consists in verifying that $N_1=N_5$ in this case, with N_1 as before and

$$N_5=\{s\geqslant 1\,|\,p^s\leqslant r_1, p^s-\{r_1\}_{p^s}\geqslant r_1-\delta_1, \text{and } \{n_1\}_{p^s}\leqslant\{r_1\}_{p^s}\}.$$

Now $s=1$ is not counted in N_1 if $p\,|\,r$ and $r-\delta>p$, since $p=p-\{r\}_p\geqslant r+1-\delta>p+1$ is impossible. If we combine this remark with (40) and the observation that $[(\delta-1)/p]=[\delta/p]$ when $p\nmid\delta$, we see that

$$\begin{aligned} N_1&=\#\{s\geqslant 2\,|\,p^{s-1}\leqslant r_1, p^{s-1}-\{r_1\}_{p^{s-1}}\geqslant r_1-\delta_1,\\ &\qquad\text{and } \{n_1\}_{p^{s-1}}\leqslant\{r_1\}_{p^{s-1}}\}.\\ &=N_5. \end{aligned}$$

Case 3(c). Here we must prove that

$$F_\delta(n,r)\sim\frac{p(n_1-r_1)}{d(n_1-r_1;1,r_1)}F_{r_1-2}(n_1,r_1-1)$$

$$\text{if } p\,|\,(n,r), p\nmid\delta, \text{ and } \delta<r<p+\delta \tag{41}$$

(with $n=pn_1$ and $r=pr_1$). By (5),

$$(d(n;1,s))_p=\frac{n_p}{\min(n_p,p^a)} \qquad \text{with } a=\left[\frac{\log s}{\log p}\right],$$

so that what we want to prove, subject to the conditions in (41), can be written, using (37), as

$$N_1=N_6+\min(\operatorname{ord}_p(n_1-r_1),b)+1 \qquad \text{with } b=\left[\frac{\log r_1}{\log p}\right], \tag{42}$$

where N_1 is defined by (39) and

$$N_6=\#\{s\geqslant 1 \mid p^s\leqslant r_1-1, p^s-\{r_1-1\}_{p^s}\geqslant 2, \text{ and } \{n_1\}_{p^s}\leqslant\{r_1-1\}_{p^s}\}.$$

Now $[(\delta-1)/p]=[\delta/p]=r_1-1$, since $p\nmid\delta$ and $r-p<\delta<r$; in Case 3(c) the conditions for N_1 can accordingly be written, using (40) and then replacing $s-1$ by s, as

$$s\geqslant 0, p^s\leqslant r_1, p^s-\{r_1\}_{p^s}\geqslant 1, \text{ and } \{n_1\}_{p^s}\leqslant\{r_1\}_{p^s}. \tag{43}$$

The conditions for N_6 are equivalent to

$$s\geqslant 1, p^s\leqslant r_1-1, p^s-\{r_1\}_{p^s}\geqslant 1, \text{ and } \{n_1\}_{p^s}\leqslant\{r_1\}_{p^s}-1, \tag{44}$$

since on the one hand $p^s-\{r_1-1\}_{p^s}\geqslant 2$ is equivalent to $p^s\nmid r_1$, therefore to $\{r_1-1\}_{p^s}=\{r_1\}_{p^s}-1$, while on the other hand the last condition in (44) implies that $p^s\nmid r_1$.

To prove (42) we calculate N_1-N_6 by counting the number of integers s which satisfy (43) but not (44). There are 3 ways in which this can occur: by taking, in (43),

(i) $s=0$, or

(ii) $p^s=r_1$, or

(iii) $\{n_1\}_{p^s}=\{r_1\}_{p^s}$.

But (i) $\Rightarrow$ (iii) since $\{n_1\}_1=\{r_1\}_1=0$; and (ii) $\Rightarrow$ (iii) since (43) with $r_1=p^s$ gives $\{n_1\}_{p^s}=\{r_1\}_{p^s}=0$. The s which satisfy (iii) and the conditions in (43) are the s such that

$$1\leqslant p^s\leqslant r_1 \qquad \text{and} \qquad n_1\equiv r_1 \pmod{p^s}$$

(the third condition in (43) is always satisfied); they are

$$\min\left(\operatorname{ord}_p(n_1-r_1)+1, \left[\frac{\log r_1}{\log p}\right]+1\right)$$

in number, as required in (42). This concludes the proof of Theorem 3.

8. Computing $E_\delta(n, r)$

We now apply the results of the preceding chapter to the calculation of $E_\delta(n, r)$ for given δ. We have 2 formulas for $E_\delta(n, r)$; (37) was used in the proof of Theorem 3, and we now return to (35). In order to simplify our calculations we bring (35) to the following form.

Proposition 12. *Let $E_\delta(n, r)$ be as in* (35). *Then, in the notation of Theorem* 3,

$$E_\delta(n, r) = \prod_{\substack{m=r+1-\delta \\ m-\{r\}_m \geqslant r+1-\delta}}^{[(1/2)r]} \lambda(m)^{e_m(n,r)} \prod_{j=0}^{[(1/2)(\delta-1)]} \lambda(r-j)^{d_j(n,r)}, \tag{45}$$

where

$$d_j(n, r) = \sum_{i=0}^{j} \varepsilon_{r-j}(n-i). \tag{46}$$

Proof. Write (35) as $E_\delta(n, r) = \Pi_1 \Pi_2$, taking $2 \leqslant m \leqslant r/2$ in Π_1 and $r/2 < m \leqslant r$ in Π_2 (with $m - \{r\}_m \geqslant r+1-\delta$ in both). The first product in (45) is equal to Π_1, since $2 \leqslant m \leqslant r/2$ and $m \geqslant r+1-\delta+\{r\}_m \geqslant r+1-\delta \geqslant 2$ in Π_1.

Now consider

$$\Pi_2 = \prod_{\substack{(r/2) < m \leqslant r \\ m - \{r\}_m \geqslant r+1-\delta}} \lambda(m)^{e_m(n,r)}. \tag{47}$$

As $m \leqslant r < 2m$ iff $\{r\}_m = r-m$, the conditions on m in (47) are equivalent to $0 \leqslant r-m < m$ and $0 \leqslant 2(r-m) \leqslant \delta - 1$. And the latter implies the former, since $\delta - 1 \leqslant r-2$. Hence the product in (47) is taken over all m such that $0 \leqslant r-m \leqslant [\frac{1}{2}(\delta-1)]$. We transform it by taking $j := r-m$ as new variable; in order to see that Π_2 is equal to the second product in (45) we must verify that $e_{r-j}(n, r) = d_j(n, r)$, as defined by (46). For this, set $m = r-j$ in (36) and observe that $\{r\}_{r-j} = j$, since $r = (r-j)+j$ and $0 \leqslant j < r-j$ (because $j \leqslant \frac{1}{2}(\delta-1) \leqslant \frac{1}{2}r - 1 < \frac{1}{2}r$). This completes the proof.

Remark. The first product in (45) is empty, if $r+1-\delta > [\frac{1}{2}r]$:

$$\Pi_1 = 1 \qquad \text{if } [\tfrac{1}{2}(r+1)] > \delta - 1, \tag{48}$$

in the notation used in the proof.

The following examples are easily worked out using Proposition 12 and (48); remember that $r \geqslant \delta + 1$ always.

For $\delta = 1$ and $r \geqslant 2$, and for $\delta = 2$ and $r \geqslant 3$,

$$\begin{aligned} E_\delta(n, r) = \lambda(r)^{\varepsilon_r(n)} &= p \qquad \text{if } r = p^k \text{ and } p^k \mid n, \\ &= 1 \qquad \text{otherwise.} \end{aligned}$$

When $\delta = 5$,

$$\begin{aligned} E_5(n, r) &= 2^{2\varepsilon_2(n) + \varepsilon_4(n-1)} 3^{\varepsilon_3(n)} 5^{\varepsilon_5(n(n-1))} && \text{if } r = 6, \\ &= 5^{\varepsilon_5(n(n-1)(n-2))} 7^{\varepsilon_7(n)} && \text{if } r = 7, \\ &= 2^{\varepsilon_4(n) + \varepsilon_8(n)} 7^{\varepsilon_7(n(n-1))} && \text{if } r = 8, \\ &= \lambda(r)^{\varepsilon_r(n)} \lambda(r-1)^{\varepsilon_{r-1}(n(n-1))} \\ &= \lambda(r-2)^{\varepsilon_{r-2}(n(n-1)) + \varepsilon_{r-2}(n-2)} && \text{if } r \geqslant 9. \end{aligned}$$

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